SPECTRUM OF PERTURBATIONS AND STABILITY OF CONVECTIVE MOTION BETWEEN VERTICAL PLANES

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In a previous paper [1] we have investigated the behavior of small perturbations in a convective flow of a viscous incompressible fluid between two infinite, vertical, parallel planes, heated to different temperatures. Spectra of decrements of normal perturbations for small values of the Grasshof number G were constructed and the intersections of the lower levels of spectrum were investigated. The present paper uses the Galerkin method of solution with a large number of base functions. Numerical computations performed on a digital computer enabled us to find spectra of decrements over the following range of Grasshof numbers: $0 < kG < 2500$ (here k is the perturbation wave number). Deductions concerning the presence of monotonous instability in the steady flow are confirmed and the absence of oscillatory instability over the given interval of kG is shown. Finally, the form of velocity and temperature perturbations is studied.

1. Let a steady flow with velocity and temperature profiles [2]

$$v_0 = {}^1/_6\ (x^3 - x), \qquad T_0 = -x \tag{1.1}$$

be established in an infinite vertical layer of a viscous incompressible fluid, bounded by two planes $x = \pm L$, between which there exists a temperature difference equal to 2θ.

Consider small normal perturbations proportional to $\exp(-\lambda t + ikz)$ where λ is a complex decrement, k is a real wave number and z is the vertical coordinate. For dimensionless amplitudes of perturbations, equations of free convection yield

$$Q_1(T, \Phi) \equiv P^{-1}\Delta T + \lambda T + a\,(T_0'\Phi - v_0 T) = 0$$

$$Q_2(T, \Phi) \equiv \Delta^2\Phi + \lambda\Delta\Phi + T' + aH\Phi = 0$$

$$\left(G = \frac{g\beta\theta L^3}{\nu^2},\ \ P = \frac{\nu}{\chi},\ \ a = ikG,\ \ \Delta = \frac{d^2}{dx^2} - k^2,\ H = v_0'' - v_0\Delta\right) \tag{1.2}$$

with boundary conditions

$$T(\pm 1) = \Phi\ (\pm 1) = \Phi'(\pm 1) = 0 \tag{1.3}$$

where Φ and T are the respective amplitudes of perturbations of the stream function and the temperature, G is the Grasshof number, P is the Prandtl number and a prime denotes differentiation with respect to x. The magnitudes L, L^2/ν, θ and $g\beta\theta L^2/\nu$ will denote the units of distance, time, temperature and velocity, respectively.

Boundary value problem (1.2) and (1.3) is solved by the Galerkin method. Approximate solution of this problem is sought in the form of Expansions

$$T^* = \sum_{m=0}^{p-1} \alpha_m T_m^{(0)}, \qquad \Phi^* = \sum_{n=0}^{q-1} \beta_n \varphi_n^{(0)} \tag{1.4}$$

Base functions $T_m^{(0)}$ and $\varphi_n^{(0)}$ [1] satisfy Eqs.

$$P^{-1}\Delta T_m^{(0)} + \nu_m^{(0)} T_m^{(0)} = 0, \qquad \Delta^2\varphi_n^{(0)} + \mu_n^{(0)}\Delta\varphi_n^{(0)} = 0 \tag{1.5}$$

Requirement that $Q_1(T^*, \Phi^*)$ and $Q_2(T^*, \Phi^*)$ are orthogonal to the functions $\{T_r^{(0)}\}$ $(r = 0, 1, 2, \ldots, p-1)$ and $\{\varphi_s^{(0)}\}$ $(s = 0, 1, \ldots, q-1)$ respectively, leads to a system of linear homogeneous algebraic equations for the coefficients of expansion (1.4)

$$\sum_{m=0}^{p-1} \alpha_m [(\nu_m^{(0)} - \lambda)\,\delta_{rm} + aB_{rm}] + a\sum_{n=0}^{q-1} \beta_n C_{rn} = 0 \qquad (r = 0, 1, 2, \ldots, p-1)$$

$$-\sum_{m=0}^{p-1} \alpha_m D_{ms} + \sum_{n=0}^{q-1} \beta_n [(\mu_n^{(0)} - \lambda)\,\delta_{sn} - aH_{sn}] = 0 \qquad (s = 0, 1, 2, \ldots, q-1) \tag{1.6}$$

Matrix elements B_{rm}, C_{rn}, D_{ms} and H_{sn} are defined as follows:

$$B_{rm} = \frac{1}{Y_r}\int_{-1}^{1} T_r^{(0)} v_0 T_m^{(0)}\, dx = \frac{4\rho_r\rho_m}{3Y_r(\rho_r^2 - \rho_m^2)^4}\,[6(\rho_r^2 + \rho_m^2) - (\rho_r^2 - \rho_m^2)^2]\,(-1)^{1/2(r+m+1)}$$

$$(r = 2\nu,\ m = 2\nu + 1) \quad \text{for} \quad (r = 2\nu + 1,\ m = 2\nu)$$

$$C_{rn} = -\frac{1}{Y_r}\int_{-1}^{1} T_r^{(0)} T_0' \varphi_n^{(0)}\, dx = \begin{cases} F_{rn} Y_r^{-1}(-1)^{(r+2)/2} & (r, n = 2\nu) \\ F_{rn} Y_r^{-1}(-1)^{(r+1)/2} & (r, n = 2\nu + 1) \end{cases}$$

$$D_{ms} = \frac{1}{J_s}\int_{-1}^{1} T_m^{(0)\prime} \varphi_s^{(0)}\, dx = \begin{cases} F_{ms} J_s^{-1} k \tanh k\,(-1)^{(m-1)/2} & (m = 2\nu + 1,\ s = 2\nu) \\ F_{ms} J_s^{-1} k \coth k\,(-1)^{m/2} & (m = 2\nu,\ s = 2\nu + 1) \end{cases}$$

$$H_{sn} = \frac{1}{J_s}\int_{-1}^{1} \varphi_s^{(0)} H \varphi_n^{(0)}\, dx = \begin{cases} 2J_s^{-1}[K_{sn} + L_{sn} k \tanh k + M_{sn} k \coth k] & (s = 2\nu + 1,\ n = 2\nu) \\ 2J_s^{-1}[K_{sn} + L_{sn} k \coth k + M_{sn} k \tanh k] & (s = 2\nu,\ n = 2\nu + 1) \end{cases}$$

where (1.7)

$$Y_r = \int_{-1}^{1} T_r^{(0)2}\, dx = 1, \qquad \rho_m = \frac{\pi}{2}(m+1), \qquad F_{rn} = \frac{2\rho_r \mu_n^{(0)}}{\nu_r^{(0)} P(\nu_r^{(0)} P - \mu_n^{(0)})}$$

$$J_s = \int_{-1}^{1} \varphi_s^{(0)} \Delta\varphi_s^{(0)}\, dx = \begin{cases} \mu_s^{(0)}[u_s^{-1} k \tanh k\,(1 - k\tanh k) - 1] & (s = 2\nu) \\ \mu_s^{(0)}[u_s^{-1} k \coth k\,(1 - k\coth k) - 1] & (s = 2\nu + 1) \end{cases}$$

$$K_{sn} = g_n - \frac{1}{3} - \frac{1}{4k^2} - \frac{1}{\mu_s^{(0)}} - \frac{1}{\mu_n^{(0)}} + \frac{(\mu_s^{(0)} + \mu_n^{(0)})(3 + \mu_n^{(0)})}{3(\mu_s^{(0)} - \mu_n^{(0)})^2} - \frac{\mu_s^{(0)}\mu_n^{(0)} l_{sn} + \mu_n^{(0)2} l_{ns}}{(\mu_s^{(0)} - \mu_n^{(0)})^4}$$

$$L_{sn} = h_{sn} + \frac{1}{4k^2} - g_n - \frac{\mu_n^{(0)} l_{ns}}{(\mu_s^{(0)} - \mu_n^{(0)})^3}, \qquad M_{sn} = -h_{sn} + \frac{1}{4k^2} + \frac{3u_n - k^2}{\mu_n^{(0)2}} + \frac{\mu_n^{(0)} l_{sn}}{(\mu_s^{(0)} - \mu_n^{(0)})^3}$$

$$u_s = \mu_s^{(0)} - k^2, \qquad g_n = \frac{u_n - 3k^2}{\mu_n^{(0)2}}, \qquad l_{sn} = u_s + 3u_n, \qquad h_{sn} = \frac{1}{\mu_s^{(0)} - \mu_n^{(0)}} - \frac{1}{\mu_s^{(0)}} + \frac{1}{\mu_n^{(0)}}$$

Condition of existence of a nontrivial solution of the nonlinear homogeneous system (1.6) defines the spectrum of characteristic decrements λ of perturbations and their dependence on the Grasshof number G, Prandtl number P and the wave number k. Problem of determination of the spectrum is connected with establishing the eigenvalues λ of the normal matrix of order $N = p + q$, constructed from the coefficients of (1.6). This matrix can be reduced to quasitriangular form by the orthogonal power method given by Voevodin in [3] (*).

This transformation results in formation, on the main diagonal, of block matrices of the order smaller than N. These can be expanded into polynomials in powers of λ, whose roots then give the required spectrum of eigenvalues. Eigenvector of the matrix is determined by the Gauss' method.

All computations were performed on the digital computer "Aragats" in the Computational Center of the Perm' University.

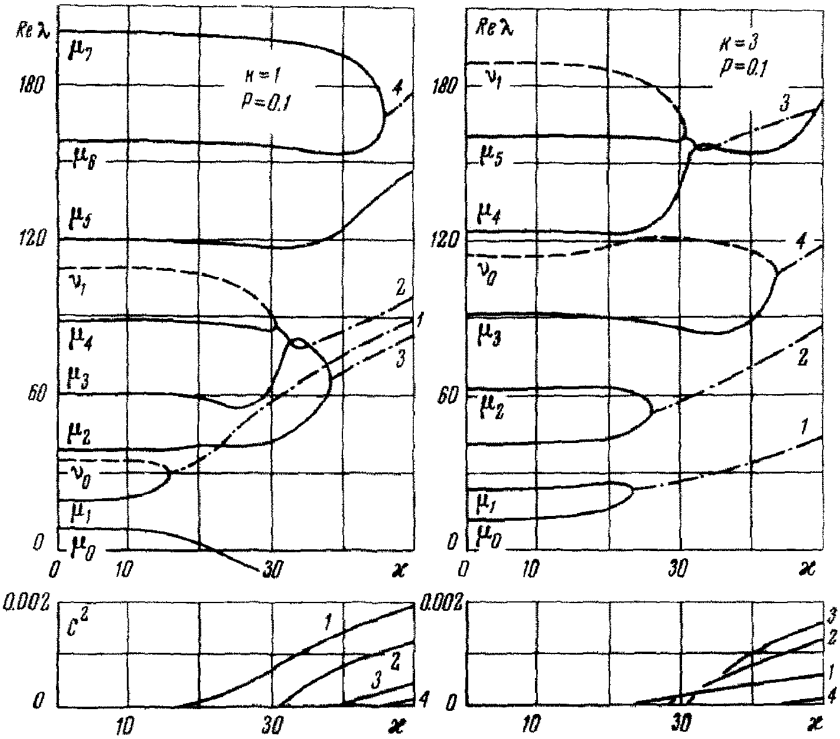

Fig. 1

2. Approximation using 28 base functions ($p = q = 14$) was utilized for the computation of decrements. Such approximation makes feasible the construction of between 9 and 14 (depending on the Prandtl number) lower levels of the spectrum of decrements over the following range of Grasshof numbers, $0 < kG < 2500$. Convergence of expansions (1.4) deteriorates with increasing G. To check the convergence, the decrements

*) This method was used earlier by Birikh [4 and 5] when investigating perturbation spectra of plane isothermal flows possessing an odd velocity profile.

were calculated using various numbers of base functions (N = 24, 26, 28) and results obtained for N = 26 and 28 were found, within the given range of kG, to be practically identical.

Figs. 1 to 3 show the dependence of the real part of the decrement Re λ and of the magnitude $c^2 = (\mathrm{Im}\lambda / kG)^2$ on the parameter $\varkappa = (kG)^{1/2}$ for perturbations with wave numbers k - 1 and 3 and for three values of the Prandtl number P = 0.1, 1 and 10. Real parts of decrements characterize the rate of decay (Re λ > 0) or growth (Re λ < 0) of perturbations, while c can be interpreted as phase velocity of perturbations in terms of velocity of the main flow.

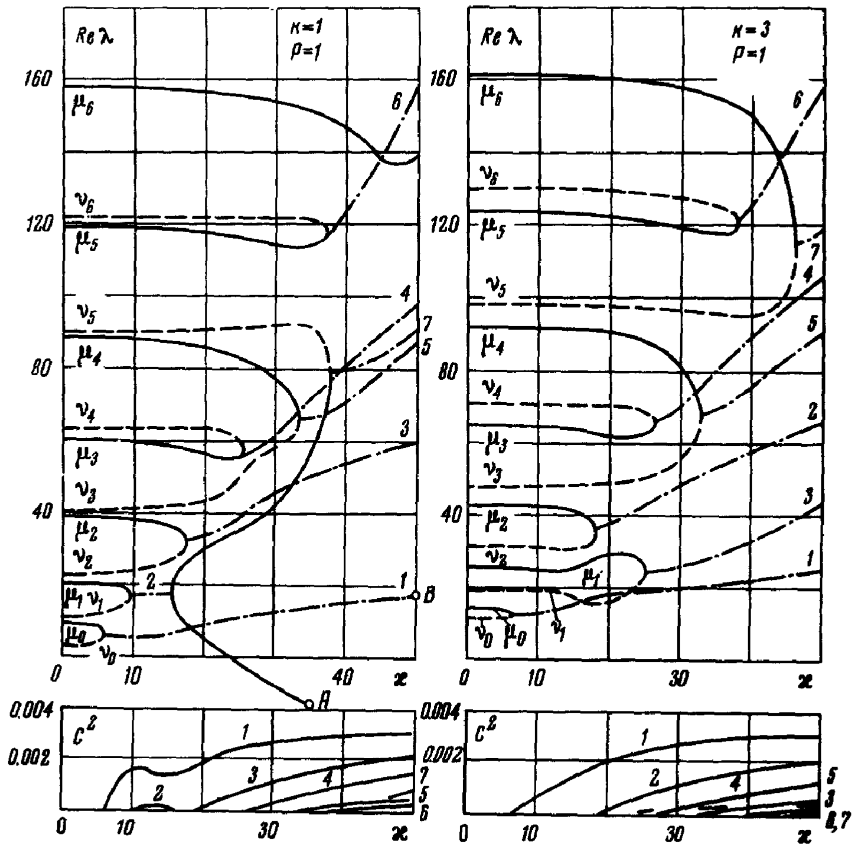

Fig. 2

In the spectra of Re λ, solid and broken lines originating on the axis kG = 0 represent, respectively, the levels of isothermal and nonisothermal perturbations, i. e. the μ- and ν-levels [1]. The dash-dot line represents real parts of the complex conjugate decrements.

We see from Figs. 1 and 3 that at small values of kG all decrements are real and positive, and the corresponding perturbations decay monotonously (*). With increasing

*) In case of degeneration of the nonperturbed spectrum (G = 0), decaying oscillatory perturbations can occur at the arbitrarily small values of G.

kG, real levels merge pairwise forming complex conjugate decrements. At sufficiently large values of kG, all perturbations in the lower part of the spectrum become oscillatory (with exception of a real level intersecting the axis $\mathrm{Re}\lambda = 0$. Singular points corresponding to transition from the monotonous to escillatory perturbations, are also peculiar for the spectra of isothermal flows with odd profiles [4 and 5]. In a large number of spectra constructed by us no evidence was found for a "simple" type of intersection of real levels, and this leads us to believe that such intersections are impossible for the given convective problem.

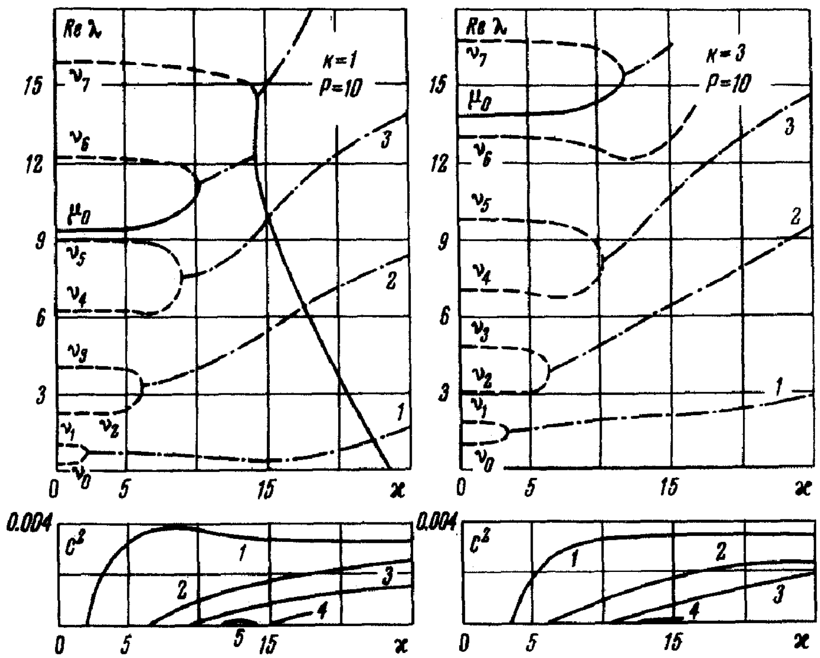

Fig. 3

A distinct feature of decremental spectra given above (as compared with the spectra of isothermal flows) is the presence in them of singular points, where a pair of complex conjugate decrements splits, as kG increases, into two real levels (Fig. 2a and 3a), the lower of which always intersects the axis $\mathrm{Re}\,\lambda = 0$ and is connected with the appearance of instability.

Shape of the decremental spectrum is basically governed by the Prandtl number. At its low values ($P = 0.1$), lower part of the spectrum (Fig. 1) contains, basically, the levels of isothermal perturbations. Lowest μ -levels differ little from the corresponding levels of the isothermal problem possessing the same velocity profile of the basic flow [5]. The most populated decremental spectrum (14 levels) is shown on Fig. 2 for $P = 1$. This case is characteristic for the present problem, as at $P \approx 1$, ν - and μ- levels alternate and the pattern of spectrum is defined, in general, by interactions between the perturbations of various types. At higher values of P (Fig. 3, $P = 10$), lower part of the spectrum is occupied by the decrements of nonisothermal perturbations. Fig. 3a shows an interesting behavior of the levels μ_0, ν_6 and ν_7 which produce, at the intersections, three singular points.

3. Analysis of the spectra of decrements leads to the following conclusions concerning the stability of the steady flow under consideration. Figs. 1 to 3, ($k = 1$) show that the axis Re $\lambda = 0$ is intersected by a real level. Point of intersection yields the critical Grasshof number G^*, at which the steady flow becomes unstable with respect to stationary perturbations. Neutral curve $G^*(k)$ has a minimum at some critical value of the wave number k_m. Fig. 4 illustrates the relationship between the lowest critical Grasshof number G^*_m and the Prandtl number P.

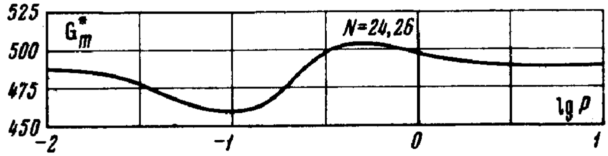

Fig. 4

We see from it that the dependence of G^*_m on P is weak. Critical wave number k_m is also weakly dependent on P; $k_m \approx 1.4$ over the range $0.01 < P < 10$.

Presence of the monotonous instability can also be deduced using a comparatively simple approximation of four base functions ($p = q = 2$) [6]. Critical values obtained in the approximations $N = 4$ and $N = 20$ fall fairly close, largest divergence obtained at small values of P, is 20%.

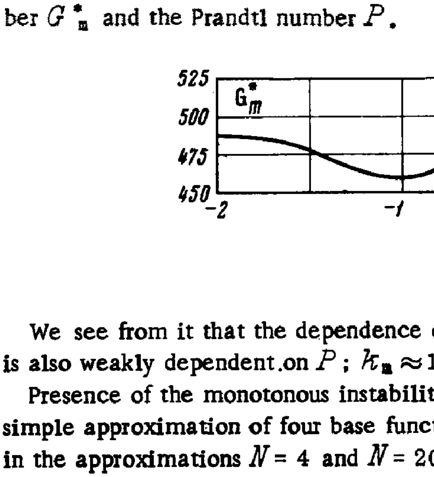

Fig. 5

Method of computation adopted here cannot be used to find the boundary of monotonous stability at large values of P. This is caused by the fact that at large P, monotonous instability depends on the first isothermal μ_0-level, below which we find (Fig. 3a) an appreciable number of nonisothermal ν-levels. Consequently, to describe the behavior of the critical level we must consider its interaction with a large number of ν-levels and this requires, at $P \gg 1$, a high order approximation. Those used by us ($p = q = 14$) make it possible to find critical Grasshof numbers G^* only for the values of P of up to $P \sim 10$.

Development of monotonous instability should, obviously, be accompanied by the appearance of secondary steady flows periodic along the layer. This was found experimentally by Elder in [7] for higher values of the Rayleigh number. Unfortunately, quantitative comparison of the theory with experiment cannot be made for two reasons. Firstly, experiments in [7] utilized a layer of finite height, consequently the appearance of secondary flows was preceded by the formation of a boundary layer, and the stream in which the instability developed has velocity and temperature profiles differing appreciably from those given in (1.1). Secondly, fluid possessing high Prandtl number ($P \sim 10^3$) was used in the experiments, while our method of computation is valid only up to $P \sim 10$.

Authors of [6] present a case for existence, at comparatively small values of Grasshof number, of oscillatory instability with respect to moving perturbations in addition to monotonous instability. Results quoted in the present paper, however, point to the absence of such instability within the given range of kG. Figs. 1 to 3 show that Re $\lambda > 0$ for all complex conjugate decrements, i. e. oscillatory perturbations decay. Fig. 5 shows two lowest levels of the spectrum Re λ ($k = 0.5$, $P = 10$) calculated for various N. We see that when $N = 4$, then the real part of λ becomes zero at some G. However, in this range of values of kG the results vary appreciably with N, and high approximation ($N \sim 20$) is needed to stabilize the values of λ. We find that when $N = 20$ or 24, real part of λ is positive over the whole range of values of kG. Thus, presence of oscillatory instability suggested in [6] is not confirmed when higher approximations are used.

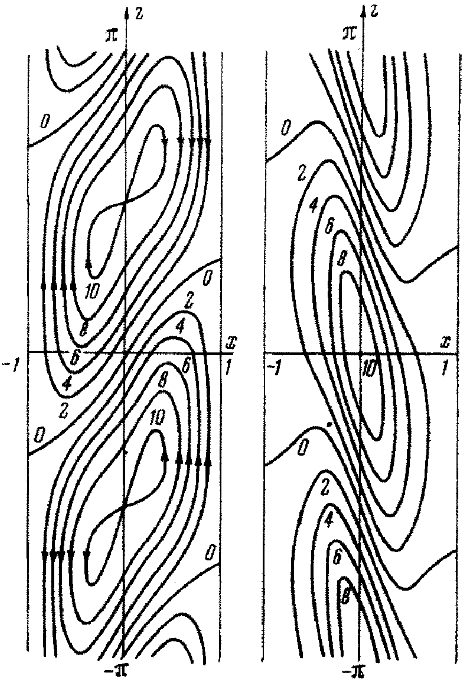

Fig. 6

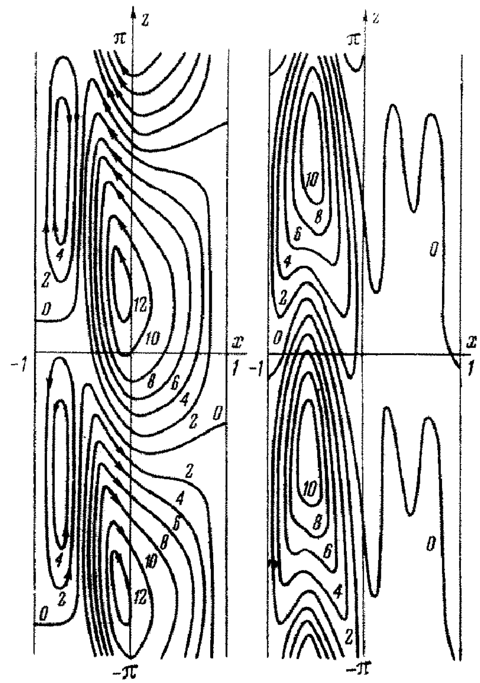

Fig. 7

Our method leaves open the problem of spectrum and, in particular, the problem of existence of oscillatory instability for $kG > 2500$. Gotoh and Satoh in [8] use an asymptotic method to show that oscillatory instability appears at very high values of Grasshof number ($G^* = 4.6 \times 10^6$).

4. Figs. 6 and 7 show the streamlines (on the left) and isotherms of characteristic perturbations with $k = 1$ and $P = 1$. Fig. 6 illustrates a monotonously increasing perturbation with the decrement $\lambda = -13.247$, when $(kG)^{\frac{1}{2}} = 55$ (point A on Fig. 2a). It is worth noting that the streamlines (Fig. 6a) practically coincide with the streamlines of a monotonously increasing perturbation in an isothermal flow possessing the same velocity profile [5]. Fig. 7 shows a decaying oscillatory perturbation with a decrement $\lambda = 17.727 + 141.29i$, when $\varkappa = (kG)^{\frac{1}{2}} = 50$ (point B on Fig. 2a). This is cancelled by the flow in the positive direction of the z-axis.

Constriction of the streamlines and isotherms in the left-hand side of the channel points to the fact that the main part of the perturbations is concentrated in that half of the channel, in which the direction of the drift coincides with the direction of velocity vector of the steady flow.

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